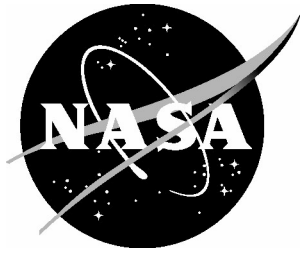


NASA/TM-2007-214876



Quasi-Static Analysis of Round LaRC THUNDER Actuators

Joel F. Campbell
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May 2007

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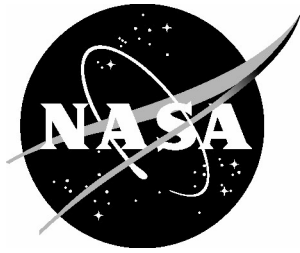
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NASA/TM-2007-214876



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May 2007

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Foreword

This report represents work completed in the 1997-1998 time frame while working as a NASA contractor for SAIC at NASA Langley Research Center. It was originally submitted as a paper to AIAA in March., 1998. This was later withheld by the author because it was simultaneously submitted as part of a patent disclosure, LAR-15827, in April, 1998.

Much has changed since the time this report was written. This model has since been improved upon over the past 9 years since it was written. It is being reproduced here for historical purposes and to make it more referenceable for those who may wish to reference it.

Quasi-Static Analysis of Round LaRC Thunder Actuators

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Abstract

An analytic approach is developed to predict the shape and displacement with voltage in the quasi-static limit of round LaRC Thunder Actuators. The problem is treated with classical lamination theory and Von Karman non-linear analysis. In the case of classical lamination theory exact analytic solutions are found. It is shown that classical lamination theory is insufficient to describe the physical situation for large actuators but is sufficient for very small actuators. Numerical results are presented for the non-linear analysis and compared with experimental measurements. Snap-through behavior, bifurcation, and stability are presented and discussed.

Introduction

Thunder actuators are devices constructed from an isotropic laminate of aluminum, LaRC SI adhesive, piezo-electric PZT, and a metal backing of either steel or brass. The LaRC adhesive is a 1 mil thick solid thermoplastic and the aluminum layer is indented to provide electrical conductivity after the actuator is cured at 250 deg C. After the curing process they form a bow shape after reaching room temperature. If simply supported or arranged in a clamshell arrangement they produce a linear motion when an external voltage is applied.

A simple method for controlling the manufacturing process and predicting the actuation amplitude has long been sought for Thunder actuators. This has been recently been treated by Campbell[3,4] in his treatment of rectangular actuators. However, the round actuators are particularly important in the manufacture of pumps and speakers. Because of the non-linear nature of the problem, the solution is not necessarily a dome. Unlike the case with the rectangular actuators, this is a highly undesirable effect. What is needed is a method for predicting the shape before the device is manufactured.

One approach to this problem is to use classical lamination theory. This is a very old subject based on the theory of composite laminates. With laminates, temperature effects are very important. As a result a very rich theory has been developed to minimize these temperature effects. Although the theory is sufficient to find an optimal lay-up to

minimize temperature effects, it is quite deficient in predicting the actual shape in all cases. The reason for this is because it is a linear approximation to a problem that is essentially non-linear. As a result, a non-linear correction to the theory is needed to describe large actuators.

The type of correction used here is one that is valid for large displacement and small strain. This correction is usually attributed to Von Karman [2] in plate theory. Campbell[3,4] has applied this to rectangular actuators with good success. This work was inspired by Hyre[5] who used this technique in his treatment of the temperature effects of orthotropic laminates. He used this in combination with a simple Raleigh-Ritz energy method to obtain solutions exhibiting the snap-through behavior. Isotropic laminates produce domes that approach cylinders when the laminate is large. It is possible to treat the round actuator case in exactly the same way as the rectangular case by converting to cylindrical coordinates before calculating the energy.

Solution by Classical Lamination Theory

Derivation of equations

Classical lamination theory [1] assumes strains are small, and amplitudes are shallow. In addition, it assumes both shearing and extensional strains perpendicular to the surface are zero. These assumptions follow the Kirchhoff-Love hypothesis for shells.

Under deformation the thin plate is described by three variables - u , v and w , where u is the displacement in x direction, v in the y and w is the height of the plate in the z direction. The variables u , v and w are themselves functions of x , y and z . It is assumed u and v follow the relation,

$$\begin{aligned} u &= u_0 - z \frac{\partial w_0}{\partial x}, \\ v &= v_0 - z \frac{\partial w_0}{\partial y}, \end{aligned} \tag{1}$$

where the 0 subscript denote the quantity is measured about the middle surface. The strains are given by [2],

$$\begin{aligned}
\varepsilon_x &= \sqrt{1 + 2 \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \approx \frac{\partial u}{\partial x}, \\
\varepsilon_y &= \sqrt{1 + 2 \frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} - 1 \approx \frac{\partial v}{\partial y}, \\
\gamma_{xy} &= \sin^{-1} \left(\frac{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}}{(1 + \varepsilon_x)(1 + \varepsilon_y)} \right) \approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{aligned} \tag{2}$$

The results of Equation 1 and Equation 2 imply,

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2}, \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2}, \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w_0}{\partial x \partial y}.
\end{aligned} \tag{3}$$

The above implies the relationship,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + z \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}, \tag{4}$$

where κ_x , κ_y , and κ_{xy} are the x, y and twist curvatures. The middle surface strains and curvatures are given by,

$$\begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{pmatrix}, \quad (5)$$

$$\begin{pmatrix} \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{pmatrix} = - \begin{pmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial^2 w_0}{\partial x \partial y} \end{pmatrix}.$$

The relationship between stress and strain is given by,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \bar{\mathbf{Q}} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}, \quad (6)$$

where \mathbf{Q} is the reduced stiffness and $\bar{\mathbf{Q}}$ is the transformed reduced stiffness. For isotropic media,

$$\bar{\mathbf{Q}} = \mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{pmatrix} = \begin{pmatrix} \frac{Y}{1-\nu^2} & \frac{\nu Y}{1-\nu^2} & 0 \\ \frac{\nu Y}{1-\nu^2} & \frac{Y}{1-\nu^2} & 0 \\ 0 & 0 & \frac{Y}{2(1+\nu)} \end{pmatrix}, \quad (7)$$

where Y is Young's modulus and ν is Poisson's ratio. To find the force and moments we first find the stresses for the k 'th layer by,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}_k = (\mathbf{Q})_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}. \quad (8)$$

Now we use the relationship,

$$\begin{pmatrix} N_x^0 \\ N_y^0 \\ N_{xy}^0 \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} dz, \quad (9)$$

$$\begin{pmatrix} M_x^0 \\ M_y^0 \\ M_{xy}^0 \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} z dz,$$

where h is the thickness. By Equation 9 and 8 we find,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + \mathbf{B} \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}, \quad (10)$$

$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \mathbf{B} \begin{pmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + \mathbf{D} \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix},$$

where \mathbf{A} , \mathbf{B} , and \mathbf{D} are the extensional, coupling and bending stiffness matrices and are given by,

$$A_{ij} = \sum_{k=1}^N (Q_{ij})_k (z_k - z_{k-1}),$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N (Q_{ij})_k (z_k^2 - z_{k-1}^2), \quad (11)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N (Q_{ij})_k (z_k^3 - z_{k-1}^3),$$

where the z 's are measured with respect to $z=0$ (at a point half way through the thickness). Since each layer is isotropic the symmetry of the stiffness matrices will be the same as the reduced stiffness matrix. Moreover, since in this case the layers are not symmetric about $z=0$ the coupling stiffness will be non-zero so that,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & A_1 & 0 \\ 0 & 0 & A_3 \end{pmatrix}, \\
\mathbf{B} &= \begin{pmatrix} B_1 & B_2 & 0 \\ B_2 & B_1 & 0 \\ 0 & 0 & B_3 \end{pmatrix}, \\
\mathbf{D} &= \begin{pmatrix} D_1 & D_2 & 0 \\ D_2 & D_1 & 0 \\ 0 & 0 & D_3 \end{pmatrix}
\end{aligned} \tag{12}$$

We now have everything we need to find the thermal loading and actuation loading so we may solve the problem.

Thermal Loading

The thermal loading takes two forms. The first is extensional and the second is bending. The problem is greatly simplified since we have only isotropic layers with equal thermal strains in each direction. The thermal loading may be calculated by,

$$\begin{aligned}
\begin{pmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{pmatrix} &= \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \Delta T \, dz \equiv \begin{pmatrix} N_x^T \\ N_y^T \\ 0 \end{pmatrix}, \\
\begin{pmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{pmatrix} &= \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \Delta T \, z \, dz \equiv \begin{pmatrix} M_x^T \\ M_y^T \\ 0 \end{pmatrix},
\end{aligned} \tag{13}$$

where,

$$\begin{aligned}
N^T &= \Delta T \sum_{k=1}^N \alpha_k (\mathcal{Q}_1 + \mathcal{Q}_2)_k (z_k - z_{k-1}), \\
M^T &= \frac{\Delta T}{2} \sum_{k=1}^N \alpha_k (\mathcal{Q}_1 + \mathcal{Q}_2)_k (z_k^2 - z_{k-1}^2).
\end{aligned} \tag{14}$$

Actuation Loading

By analogy with thermal loading,

$$\begin{pmatrix} N_x^A \\ N_y^A \\ N_{xy}^A \end{pmatrix} = \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} d_{31} \\ d_{31} \\ 0 \end{pmatrix} \frac{V}{t} dz \equiv \begin{pmatrix} N^A \\ N^A \\ 0 \end{pmatrix}. \quad (15)$$

$$\begin{pmatrix} M_x^A \\ M_y^A \\ M_{xy}^A \end{pmatrix} = \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} d_{31} \\ d_{31} \\ 0 \end{pmatrix} \frac{V}{t} z dz \equiv \begin{pmatrix} M^A \\ M^A \\ 0 \end{pmatrix},$$

where t is the thickness of the PZT, V is the applied voltage, d_{31} is the actuation constant in a plane perpendicular to the poling axis and,

$$N^A = d_{31} (Q_1 + Q_2)_3 V, \quad (16)$$

$$M^A = d_{31} (Q_1 + Q_2)_3 (z_3 + z_2) V.$$

Here it is assumed we only have actuation on the third layer.

Thermal and Actuation Displacement

To find the resulting strains and curvatures in terms of the actuation loading we first add both the thermal and actuation sources as,

$$\begin{pmatrix} N_x' \\ N_y' \\ N_{xy}' \end{pmatrix} \equiv \begin{pmatrix} N^A + N^T \\ N^A + N^T \\ 0 \end{pmatrix} = \begin{pmatrix} N' \\ N' \\ 0 \end{pmatrix}, \quad (17)$$

$$\begin{pmatrix} M_x' \\ M_y' \\ M_{xy}' \end{pmatrix} \equiv \begin{pmatrix} M^A + M^T \\ M^A + M^T \\ 0 \end{pmatrix} = \begin{pmatrix} M' \\ M' \\ 0 \end{pmatrix}.$$

Now we use the results of Equation 10 so that,

$$\begin{pmatrix} N \\ N \\ 0 \\ M \\ M \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}. \quad (18)$$

Inversion of the above gives,

$$\begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{pmatrix}^{-1} \begin{pmatrix} N \\ N \\ 0 \\ M \\ M \\ 0 \end{pmatrix}. \quad (19)$$

By using the results of 12 and inverting the 6x6 matrix we find,

$$\begin{aligned} \varepsilon_x^0 \equiv \varepsilon_y^0 &= \frac{(D_1 + D_2) N - (B_1 + B_2) M}{(A_1 + A_2)(D_1 + D_2) - (B_1 + B_2)^2}, \\ \kappa_x \equiv \kappa_y &= \frac{(A_1 + A_2) M - (B_1 + B_2) N}{(A_1 + A_2)(D_1 + D_2) - (B_1 + B_2)^2}, \\ \gamma_{xy} &= \kappa_{xy} = 0. \end{aligned} \quad (20)$$

So we have thus found the curvature in terms of known quantities. The shape is that of a dome. The simplicity of the relationship comes about because the layers are isotropic. The solution is valid for small actuators such as the round ones used for pumps and speakers.

Solution by the Von Karman Non-Linear Approximation

The Von Karman approximation assumes large displacement and small strain. This suggests including second order terms in w in addition to the linear terms for u , v and w . Expanding Equation 2 to include these terms give,

$$\begin{aligned}
\varepsilon_x^0 &\approx \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \\
\varepsilon_y^0 &\approx \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \\
\gamma_{xy}^0 &\approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}.
\end{aligned} \tag{21}$$

This combined with the results of Equation 1 give,

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial^2 w_0}{\partial x^2}, \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 - z \frac{\partial^2 w_0}{\partial y^2}, \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} - 2z \frac{\partial^2 w_0}{\partial x \partial y}.
\end{aligned} \tag{22}$$

The total energy takes the form,

$$U = \iiint (dU_0 - dW_T - dW_A), \tag{23}$$

where dU_1 is the stored elastic energy volume density and dW_T and dW_A are the temperature and actuation contributions. The elastic energy portion of it for the k 'th layer is,

$$\begin{aligned}
(dU_0)_k &= \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}), \\
&= \left(\frac{1}{2} Q_1 \varepsilon_x^2 + Q_2 \varepsilon_x \varepsilon_y + \frac{1}{2} Q_1 \varepsilon_y^2 + \frac{1}{2} Q_3 \gamma_{xy}^2 \right)_k.
\end{aligned} \tag{24}$$

For the thermal portion we have,

$$\begin{aligned}
(dW_T)_k &= (\sigma_x^T \varepsilon_x + \sigma_y^T \varepsilon_y)_k, \\
&= \left((Q_1 + Q_2) \alpha \Delta T \varepsilon_x + (Q_1 + Q_2) \alpha \Delta T \varepsilon_y \right)_k.
\end{aligned} \tag{25}$$

For the actuation portion we have,

$$\begin{aligned} (dW_A)_3 &= (\sigma_x^A \epsilon_x + \sigma_y^A \epsilon_y)_3, \\ &= (Q_1 + Q_2)_3 d_{31} \frac{V}{t} \epsilon_x + (Q_1 + Q_2)_3 d_{31} \frac{V}{t} \epsilon_y \end{aligned} \quad (26)$$

Now that we have the general form for the energy integral we may try a Raleigh-Ritz type solution. The method involves making a guess for the solution. This guess should look qualitatively like the expected solution yet be flexible enough to allow for adjustment. This adjustment comes in the form of changing the guess based on a finite number of parameters which the guess contains. The parameters are adjusted in such a way as to minimize the total energy of the system. In this situation it would be advantageous to choose a guess that is close enough to the classical lamination result that in the linear limit (small scaling) the solution approaches the classical lamination result. Such a guess takes the form,

$$w^0 = \frac{1}{2} (a x^2 + b y^2). \quad (27)$$

Classical lamination theory assumes no shearing strain between layers for thermal expansion. If we make the same assumptions here we find u^0 and v^0 must take the form,

$$\begin{aligned} u^0 &= c x - \frac{1}{6} a^2 x^3 - \frac{1}{4} a b x y^2, \\ v^0 &= d y - \frac{1}{6} a^2 y^3 - \frac{1}{4} a b x^2 y. \end{aligned} \quad (28)$$

The ansatz represented by Equation 27 and 28 has been used successfully by Campbell[3,4] in his treatment rectangular actuators and Hyer [5] in his treatment of orthotropic laminates. Equations 22, 27 and 28 give,

$$\begin{aligned} \epsilon_x &= c - \frac{1}{4} a b y^2 - a z, \\ \epsilon_y &= d - \frac{1}{4} a b x^2 - b z, \\ \gamma_{xy} &= 0. \end{aligned} \quad (29)$$

All that is needed now is to use Equation 23 with Equations 24, 25 and 26 and 29 to find the total energy. To do the round actuators, however, we must first convert to cylindrical coordinates. The transformation is,

$$\begin{aligned}
x &= r \cos (\theta + \varphi) \\
y &= r \sin (\theta + \varphi)
\end{aligned}
\tag{30}$$

$$dx \, dy \, dz = r \, dr \, d\theta \, dz$$

The angle φ is an arbitrary phase angle that comes about because of symmetry arguments. The total energy is obtained by integrating r from 0 to R , and θ from 0 to 2π . z is integrated through the thickness. The solution is obtained by minimizing the total energy with respect to a , b , c and d . The result of the minimization is,

$$\begin{aligned}
D_1 a + K_4 b - B_1 c - B_2 d + K_1 b^2 + 2 K_1 a b - K_2 b c - K_2 b d + \\
K_3 a b^2 + M' &= 0 \\
K_4 a + D_1 b - B_2 c - B_1 d + K_1 a^2 + 2 K_1 a b - K_2 a c - K_2 a d + \\
K_3 a^2 b + M' &= 0 \\
- B_1 a - B_2 b + A_1 c + A_2 d - K_2 a b - N' &= 0 \\
- B_2 a - B_1 b + A_2 c + A_1 d - K_2 a b - N' &= 0,
\end{aligned}
\tag{31}$$

where,

$$\begin{aligned}
K_1 &= \frac{1}{16} (B_1 + B_2) R^2 \\
K_2 &= \frac{1}{16} (A_1 + A_2) R^2 \\
K_3 &= \frac{1}{192} (3 A_1 + A_2) R^4 \\
K_4 &= D_2 + \frac{N'}{8} R^2
\end{aligned}
\tag{32}$$

The A 's, B 's and D 's have the same meaning as before and N' and M' is the same as well. The radius R is the length and width of the actuator. It is a simple matter to show that the solution in the linear limit (small scaling - $R=0$) matches the classical lamination result if we take $a=-\kappa_X$, $b=-\kappa_Y$, $c=\epsilon_X$ and $d=\epsilon_Y$. The non-linear equations are coupled and third order so it is unlikely that an exact analytic solution exists. It may be possible to find close approximate solutions to them, however. In any event they may be solved numerically with no difficulty.

Unlike the classical lamination result the non-linear analysis predicts unequal curvatures that depend on the magnitude of the scaling and the aspect ratio. It also predicts multiple possible solutions. The nature of these solutions are dome-like solutions which approach cylinders in the limit of large scaling.

As a comparison for a particular case we compute a simple example. For this case we choose a 3 layer actuator with 1 mil LaRC Si, 6.8 mil PZT and 2 mil brass with $R=.6$ in. The results are shown in Figures 1 and 2. Although we have two solutions, because there is also an arbitrary phase angle, ϕ , these are actually the same solution rotated by 90 degrees. In fact, any rotation of one of those solutions is also a solution. If R is small enough we have a single dome solution. The solutions shown in Figures 1 and 2 are sometimes referred to as a saddle mode but technically they are not true saddles since the curvatures in each direction have the same sign. As we shall see there are some cases where we can actually have 3 stable solutions. This is different from the perfectly square actuator where there are at most 2 stable solutions.

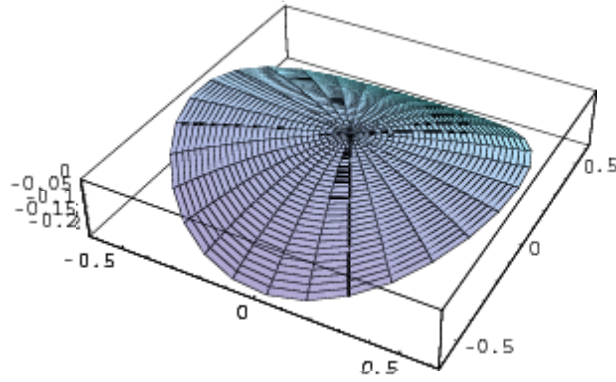


Figure 1. Non-linear Solution #1.

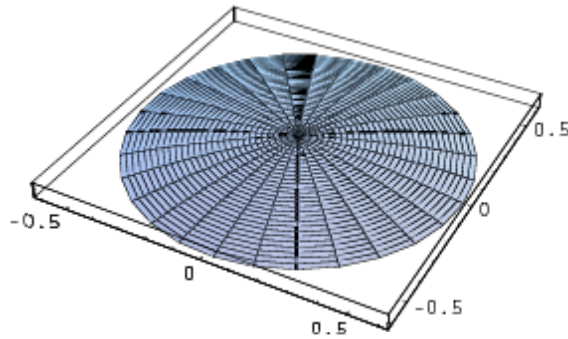


Figure 2. Non-linear Solution #2.

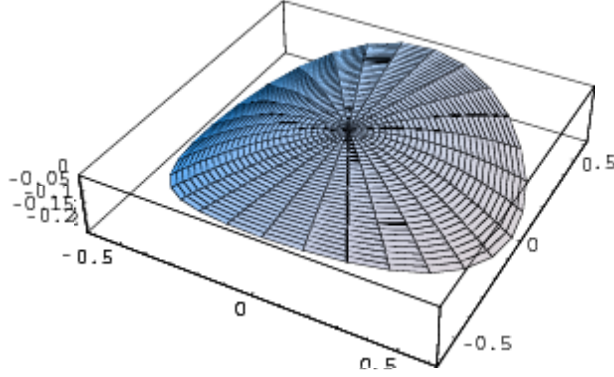


Figure 3. Non-linear Solution #3.

Stability

The stability of each solution is determined by whether or not the solutions are maxima or minima. If a particular solution represents a minimum then it is stable. If it is a maximum, it is unstable. The total energy of the system may be viewed as a function of several variables. As such, once an extremum is found, the test whether it represents minuma is determined by whether the determinant of F is positive definite where,

$$\begin{aligned} \text{Det}[F] &> 0, \\ F_{ij} &= \frac{\partial^2 U}{\partial x_i \partial x_j}. \end{aligned} \quad (32)$$

Fortunately $\partial U / \partial x_i$ is represented by the left hand side of Equation 31 (to within a multiplicative factor) so all we have to do are the x_j differentiations to find the matrix elements.

Bifurcation

As stated earlier, The solution may take on different shapes depending on the scaling. To demonstrate this, numerical results were completed for a square piece having the same material properties as before except we vary the size from $R=0$ to $R=2$ in. For a small piece the shape is a dome as predicted by classical lamination theory. At some critical size the shape turns suddenly into a dome with unequal curvatures. In the limit of large

scaling it becomes a cylinder. Figure 3 is a plot of both the x and y curvatures. In this case the critical size is at approximately 1 inch square.

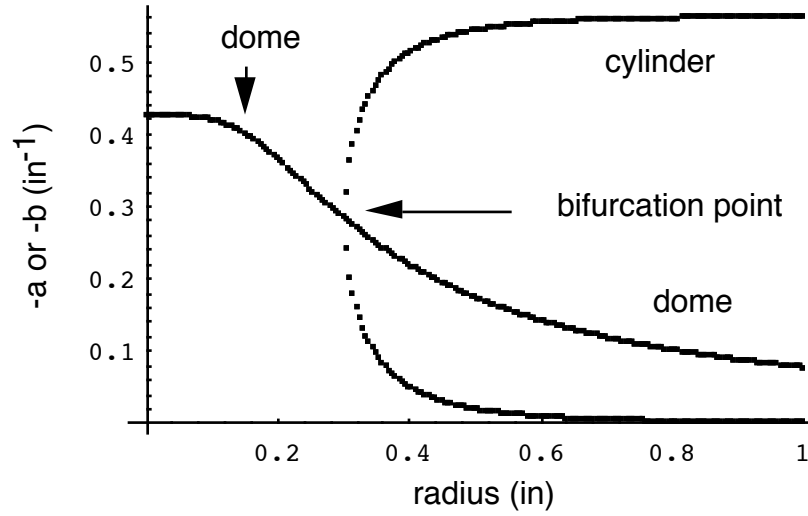


Figure 3. Bifurcation plot of x and y curvature as a function of size for a round actuator.

This result shows that there are 3 possible solutions past the critical size to thickness ratio. Experiment shows at a large enough size the perfect doming is rarely seen in Thunder actuators. We feel this is due to slipping during cure - that there is a slight preferential direction in the strain. In Rainbow actuators, on the other hand, the doming solution is seen at all sizes. In the case of Rainbow slipping is not possible because the base layer is created by a chemical reducing process.

Perfect doming solution

A single equation may be derived for the perfect doming case. This is obtained by solving Equation 31 in the simple case where $a=b$ and $d=c$. The result is,

$$\frac{R^4}{384} (3 A_1 - A_2) a^3 + \left(D_1 + D_2 - \frac{(B_1 + B_2)^2}{A_1 + A_2} \right) a + M - N \frac{B_1 + B_2}{A_1 + A_2} = 0 \quad (34)$$

Expressed in terms of the dome height this becomes,

$$\frac{1}{48 R^2} (3 A_1 - A_2) z^3 + \frac{2}{R^2} \left(D_1 + D_2 - \frac{(B_1 + B_2)^2}{A_1 + A_2} \right) z + \quad (35)$$

$$M - N \frac{B_1 + B_2}{A_1 + A_2} = 0$$

This has an exact analytic solution. It is, however, quite complicated and will not be presented here.

Deflection under non-inertial distributed load

To estimate the deflection under load we write the total energy as,

$$U = \iiint (dU_0 - dW_T - dW_A) - \int_0^R 2\pi r w P dr \quad (36)$$

where w is the height as a function of r and P is the pressure. Following the same procure as before we find,

$$\frac{R^4}{384} (3 A_1 - A_2) a^3 + \left(D_1 + D_2 - \frac{(B_1 + B_2)^2}{A_1 + A_2} \right) a + \quad (37)$$

$$M - N \frac{B_1 + B_2}{A_1 + A_2} - \frac{R^2}{8} P = 0$$

In terms of the dome height,

$$\frac{1}{48 R^2} (3 A_1 - A_2) z^3 + \frac{2}{R^2} \left(D_1 + D_2 - \frac{(B_1 + B_2)^2}{A_1 + A_2} \right) z + \quad (38)$$

$$- M + N \frac{B_1 + B_2}{A_1 + A_2} + \frac{R^2}{8} P = 0$$

This result impliies a higher stiffness than actually occurs. By taking the isotropic and linear limit and comparing with exact plate theory it can be shown this result is stiffer by about 4/3. The reason for this is because the system is heavily over constrained. The second order polynomial used here can't accurately account for the change in shape under load. For approximation purposes, however, it is sufficient.

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REPORT DOCUMENTATION PAGE					Form Approved OMB No. 0704-0188	
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1. REPORT DATE (DD-MM-YYYY) 01-05-2007		2. REPORT TYPE Technical Memorandum			3. DATES COVERED (From - To)	
4. TITLE AND SUBTITLE Quasi-Static Analysis of Round LaRC Thunder Actuators				5a. CONTRACT NUMBER		
				5b. GRANT NUMBER		
				5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Campbell, Joel F.				5d. PROJECT NUMBER		
				5e. TASK NUMBER		
				5f. WORK UNIT NUMBER 810031.07.03		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center Hampton, VA 23681-2199				8. PERFORMING ORGANIZATION REPORT NUMBER L-19357		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001				10. SPONSOR/MONITOR'S ACRONYM(S) NASA		
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) NASA/TM-2007-214876		
12. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited Subject Category 37 Availability: NASA CASI (301) 621-0390						
13. SUPPLEMENTARY NOTES An electronic version can be found at http://ntrs.nasa.gov						
14. ABSTRACT An analytic approach is developed to predict the shape and displacement with voltage in the quasi-static limit of round LaRC Thunder Actuators. The problem is treated with classical lamination theory and Von Karman non-linear analysis. In the case of classical lamination theory exact analytic solutions are found. It is shown that classical lamination theory is insufficient to describe the physical situation for large actuators but is sufficient for very small actuators. Numerical results are presented for the non-linear analysis and compared with experimental measurements. Snap-through behavior, bifurcation, and stability are presented and discussed.						
15. SUBJECT TERMS PZT; THUNDER; Actuator; Circular; Nonlinear; Piezo-electric; Rayleigh-Ritz; Round; Snap-through; Von Karman						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON	
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Help Desk (email: help@sti.nasa.gov)	
U	U	U	UU	23	19b. TELEPHONE NUMBER (Include area code) (301) 621-0390	